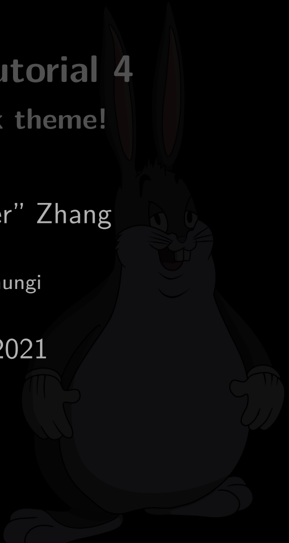
CSC363H5 Tutorial 4

this time in dark theme!



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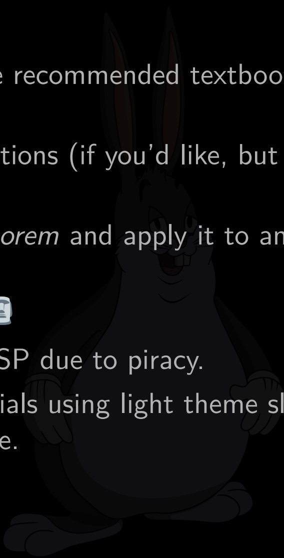
University of Chungi

February 3, 2021

Learning objectives this tutorial

By the end of this tutorial, you should...

- Find a totally legal way to obtain the recommended textbook for this course.
- Read the recommended textbook sections (if you'd like, but I think it's really worth it!).
- Be able to state the *normal form theorem* and apply it to an example problem.
- Turn yourself into a moai. 🗿 🗿 🗿
- Have your internet revoked by your ISP due to piracy.
- Appreciate the time when I ran tutorials using light theme slides instead of dark theme slides like these.

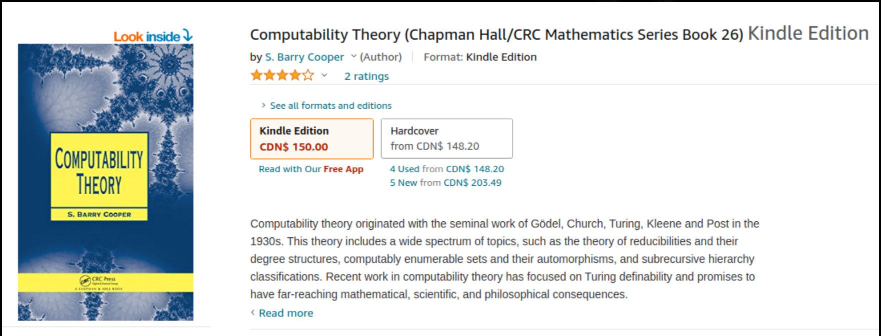


IMPORTANT NOTICE

sowwy ;-; i had to speedrun those slides, i realized i was covering the wrong content only a few hours prior. i was gonna cover some week 4 material, until i realized people in the friday lecture would probably have no idea what i'm talking about, so i had to quickly remake some slides. the point of today's tutorial is to introduce you to some proofs that we have skipped in class.

Readings? What is this, English class?

Yea, I just discovered that a (recommended) textbook exists for this course! Available for the cheap price of only $150.



yea right.

Trust me, it's worth a read! You will learn cool stuff like ~~how to solve homework problems~~ the proof of some theorems skipped in class.

Readings? What is this, English class?

Just some recommended readings[1] from me to reinforce lecture material:

- Week 2: sections 2, 4.2, 4.3 (Note the book gives a different definition of Turing machine, but it is equivalent and worth a read!)
- Week 3: sections 5.1-5.3
- Week 4: sections 5.2, 10.1 (first page)

These recommended readings are certified by `helo_fish.jpg`.

[1] not official! just what i think would be useful.

G*del is back!

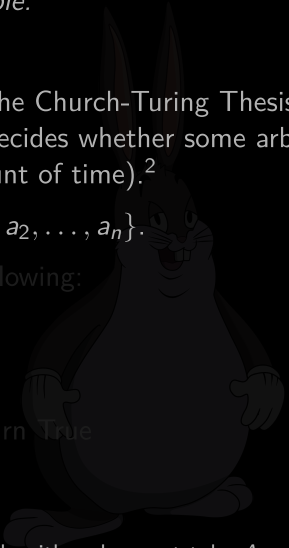
We claimed this statement is true without proving it previously:

Any finite set $A \subseteq \mathbb{N}$ is computable.

Let us prove it.

Task: Prove the above statement using the Church-Turing Thesis. In other words, describe an algorithm that decides whether some arbitrary $x \in \mathbb{N}$ is in A or not (within a finite amount of time).[2]

Hint: If A is finite, we can write $A = \{a_1, a_2, \ldots, a_n\}$.



Answer: Given an input $x \in \mathbb{N}$, do the following:

$i = 1$

while $i \leq n$:

  if $x = a_i$: return True

return False

[2]Note that A is predetermined already: your algorithm does not take A as an input.

G*del is back!

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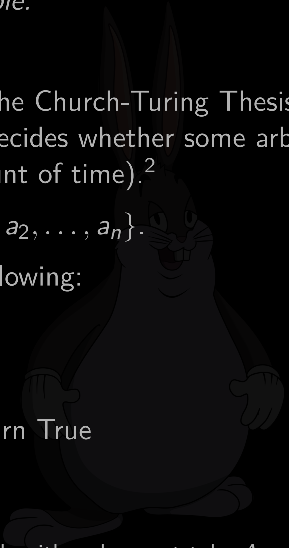


$i = 1$

while $i \leq n$:

  if $x = a_i$: return True

return False

[2] Note that A is predetermined already: your algorithm does not take A as an input.

Normal Form Theorem!

*We say a set $A \subseteq \mathbb{N}$ is **computably enumerable (c.e.)** if $A = \emptyset$ or there exists a computable $f : \mathbb{N} \to \mathbb{N}$ such that $A = \mathit{range}(f)$.*

We say a relation $R(\vec{x})$ is in Σ^0_1 if there exists a computable relation $C(a, \vec{x})$ such that for all $\vec{x}$, $R(\vec{x}) \Leftrightarrow \exists \vec{a}\, C(\vec{a}, \vec{x})$. In other words, when R is thought of as a set,

$$R = \{\vec{x} : \exists \vec{a}\, C(\vec{a}, \vec{x})\}.$$

For $e \in \mathbb{N}$, we denote by W_e the set $\mathit{dom}(\varphi_e)$, where φ_e is the partial recursive function corresponding to the e-th Turing machine.

Task: Digest those definitions.

Normal Form Theorem!

Finally, we get to the statement! :D (or D: if you don't like proofs)

Theorem

Let $A \subseteq \mathbb{N}$. The following are equivalent:

1. *A is c.e.;*
2. *$A \in \Sigma^0_1$ (when A is thought of as a unary relation);*
3. *$A = W_e$ for some $e \in \mathbb{N}$.*

We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). But before we begin, let's try some examples!

Task: Let $A = \mathbb{N}$. Show that A is c.e.. Do the same with $A = E$ (the set of even natural numbers).

Answer: For $A = \mathbb{N}$, $f(x) = x$ satisfies $\mathrm{range}(f) = A$. For $A = E$, $f(x) = 2x$ satisfies $\mathrm{range}(f) = A$.

Normal Form Theorem!

Finally, we get to the statement! :D (or D: if you don't like proofs)

Theorem

Let $A \subseteq \mathbb{N}$. The following are equivalent:

1. *A is c.e.;*
2. *$A \in \Sigma_1^0$ (when A is thought of as a unary relation);*
3. *$A = W_e$ for some $e \in \mathbb{N}$.*

We will prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). But before we begin, let's try some examples!

Task: Let $A = \mathbb{N}$. Show that A is c.e.. Do the same with $A = E$ (the set of even natural numbers).
Answer: For $A = \mathbb{N}$, $f(x) = x$ satisfies range$(f) = A$. For $A = E$, $f(x) = 2x$ satisfies range$(f) = A$.

Normal Form Theorem!

Task: Let $A = \mathbb{N}$. Show that $A \in \Sigma_1^0$ by finding a computable relation $C(a, x)$ such that for all $x \in \mathbb{N}$, $x \in A$ if and only if there exists a such that $C(a, x)$ holds. In other words,

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}.$$

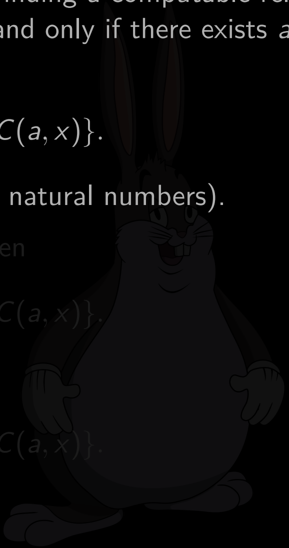
Do the same with $A = E$ (the set of even natural numbers).

Answer: For $A = \mathbb{N}$, if $C(a, x) : a = x$, then

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}.$$

For $A = E$, if $C(a, x) : 2a = x$, then

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}.$$



Normal Form Theorem!

Task: Let $A = \mathbb{N}$. Show that $A \in \Sigma^0_1$ by finding a computable relation $C(a, x)$ such that for all $x \in \mathbb{N}$, $x \in A$ if and only if there exists a such that $C(a, x)$ holds. In other words,

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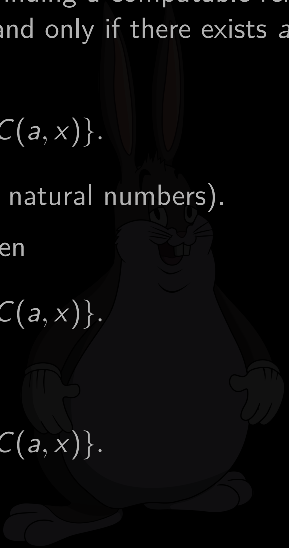
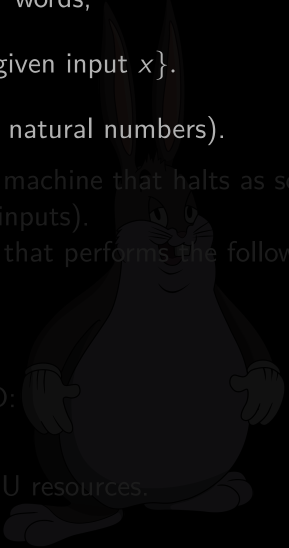


Normal Form Theorem!

Task: Let $A = \mathbb{N}$. Show that $A = W_e$ for some $e \in \mathbb{N}$, by finding a Turing machine P_e that halts only on A. In other words,

$$A = \{x \in \mathbb{N} : P_e \text{ halts given input } x\}.$$

Do the same with $A = E$ (the set of even natural numbers).



Answer: For $A = \mathbb{N}$, let P_e be the Turing machine that halts as soon as it receives any input (so that it halts on all inputs).
For $A = E$, let P_e be the Turing machine that performs the following procedure given input x:

$$\begin{aligned} &\text{if } x \bmod 2 = 0 : \\ &\quad \text{stop executing!!! D:} \\ &\text{else :} \\ &\quad \text{loop and waste CPU resources.} \end{aligned}$$

Normal Form Theorem!

Task: Let $A = \mathbb{N}$. Show that $A = W_e$ for some $e \in \mathbb{N}$, by finding a Turing machine P_e that halts only on A. In other words,

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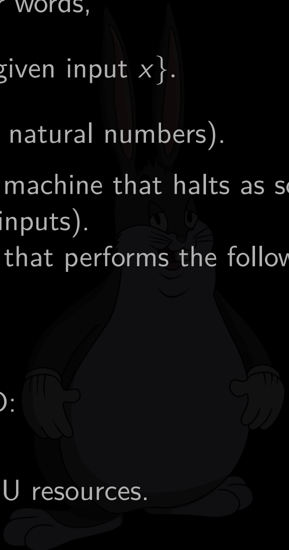


if $x \bmod 2 = 0$:

  stop executing!!! D:

else :

  loop and waste CPU resources.

Hope you like proofs.

Here's the Normal Form Theorem again.

Theorem

Let $A \subseteq \mathbb{N}$. The following are equivalent:

1. *A is c.e.;*
2. *$A \in \Sigma^0_1$ (when A is thought of as a unary relation);*
3. *$A = W_e$ for some $e \in \mathbb{N}$.*

Hopefully with the previous tasks, you have convinced yourself that the Normal Form Theorem holds for computable sets.

Hope you like proofs.

(1) $\Rightarrow$ (2): A is c.e. $\Rightarrow A \in \Sigma^0_1$.

To prove this: suppose A is c.e., so either $A = \emptyset$ or $A = \{f(0), f(1), \ldots\}$. To show $A \in \Sigma^0_1$, we want to write

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}$$

where $C(a, x)$ is a computable relation.

Task: Come up with such a computable relation $C(a, x)$ so that A satisfies the above equality.

(Hint: I think you will probably need a separate case for $A = \emptyset$).

Hope you like proofs.

To show $A \in \Sigma^0_1$, we want to write

$$A = \{x \in \mathbb{N} : \exists a\, C(a,x)\}$$

where $C(a,x)$ is a computable relation.

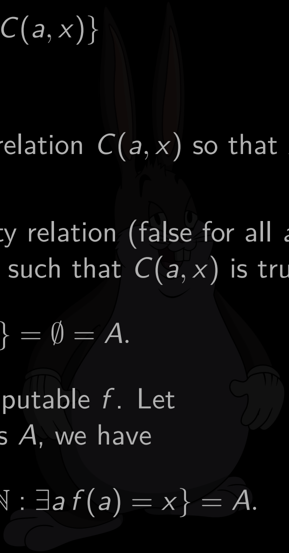
Task: Come up with such a computable relation $C(a,x)$ so that A satisfies the above equality.

Answer: If $A = \emptyset$, let $C(a,x)$ be the empty relation (false for all a, x). Then for any $x \in \mathbb{N}$, there doesn't exist a such that $C(a,x)$ is true, so

$$\{x \in \mathbb{N} : \exists a\, C(a,x)\} = \emptyset = A.$$

Otherwise $A = \{f(0), f(1), \ldots\}$ for a computable f. Let $C(a,x) : f(a) = x$. Since the range of f is A, we have

$$\{x \in \mathbb{N} : \exists a\, C(a,x)\} = \{x \in \mathbb{N} : \exists a\, f(a) = x\} = A.$$



Hope you like proofs.

(2) $\Rightarrow$ (3): $A \in \Sigma^0_1 \Rightarrow A = W_e$ for some $e \in \mathbb{N}$.

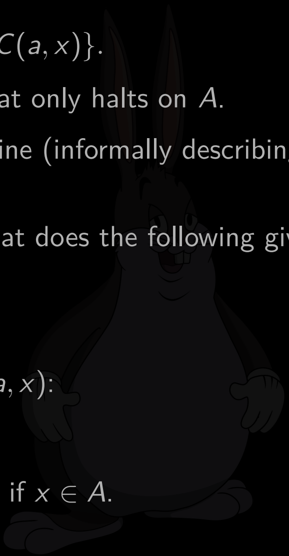
To prove this: suppose $A \in \Sigma^0_1$, so there is a computable relation $C(a, x)$ so that

$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}.$$

To show $A = W_e$ for some $e \in \mathbb{N}$, we want to produce a Turing machine that only halts on A (since W_e is the set of inputs that make the eth Turing machine halt).

Task: Come up with such a Turing machine (informally describing what it does).

Hope you like proofs.



$$A = \{x \in \mathbb{N} : \exists a\, C(a, x)\}.$$

We want to produce a Turing machine that only halts on A.

Task: Come up with such a Turing machine (informally describing what it does).

Answer: Let P_e be the Turing machine that does the following given an input $x \in \mathbb{N}$:

```
a = 0
while not C(a, x):
    a += 1
```

This program halts on input x if and only if $x \in A$.

Hope you like proofs.

(3) $\Rightarrow$ (1): $A = W_e$ for some $e \in \mathbb{N} \Rightarrow A$ is c.e.

To prove this: suppose $A = W_e$. If $A = \emptyset$, then A is c.e. by definition and we are done. Otherwise there exists a $p \in A$. We want to find a computable function f such that $A = \text{range}(f)$.

Definition: We say $\varphi_{e,s}(x)\downarrow$ when $x, e < s$ and the eth Turing machine takes s steps or less to halt on x.
Task: Digest the above definition.

Notice: $x \in W_e$ if and only if $\varphi_{e,s}(x)\downarrow$ for some $s \in \mathbb{N}$. In other words, s is large enough so that $x, e < s$ and the Turing machine takes s steps or less to halt on x.
Task: Argue that we can determine whether $\varphi_{e,s}(x)\downarrow$ or not (in a finite amount of time).

Hope you like proofs.

(3) $\Rightarrow$ (1): $A = W_e$ for some $e \in \mathbb{N} \Rightarrow A$ is c.e.

To prove this: suppose $A = W_e$. If $A = \emptyset$, then A is c.e. by definition and we are done. Otherwise there exists a $p \in A$. We want to find a computable function f such that $A = \text{range}(f)$.

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(3) $\Rightarrow$ (1): $A = W_e$ for some $e \in \mathbb{N} \Rightarrow A$ is c.e.

To prove this: suppose $A = W_e$. If $A = \emptyset$, then A is c.e. by definition and we are done. Otherwise there exists a $p \in A$. We want to find a computable function f such that $A = \text{range}(f)$.

Task: Show (or convince yourself) that the range of f is exactly A, where

$$f(\langle x, s\rangle) = \begin{cases} x & \varphi_{e,s}(x)\downarrow \\ p & \text{otherwise.} \end{cases}$$

This finishes the proof!

Task: Breathe in for 4 seconds, hold your breath for 7 seconds, then slowly release it for 8 seconds. Repeat for 3 minutes.

While doing this, imagine you are a 600-year-old giant 50-ton stone on Easter Island in the middle of the Pacific. You have access to a black box that can solve *all* problems in this universe, including non-computable problems such as deciding whether $x \in \mathbb{N}$ is in the halting set K, as it has access to any oracle in existence.
As you breath in and out, you feel your brain expanding. You are filled with *determination*. You can solve any problem. You acknowledge all the suffering and injustice you have received throughout your life, and have ascended beyond any worldly desire. A light breeze brushes up against your coarse stone surface. You feel *at peace*.
A *Gygis alba* lands on your head. You can hear it chirping amidst the light breeze. You are unbothered, because you have ascended beyond life.
You are a Moai. 🗿 🗿 🗿

🗿 🗿 🗿 🗿 🗿 🗿

Now that you have attained knowledge of this universe, you may prove the following:

> *Let $A \subseteq \mathbb{N}$. A is c.e. if and only if there exists a partial computable function f such that $A = range(f)$.*

Task: Prove the above using the normal form theorem (written below).

Theorem

Let $A \subseteq \mathbb{N}$. The following are equivalent:

1. *A is c.e.;*
2. *$A \in \Sigma^0_1$ (when A is thought of as a unary relation);*
3. *$A = W_e$ for some $e \in \mathbb{N}$.*

Hint: For proving the $\Leftarrow$ direction (which is more difficult), show A satisfies condition 2 in the above theorem. You may want to use $\varphi_{e,s}(x)\downarrow$ somewhere in this direction.

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Now that you have attained knowledge of this universe, you may prove the following:

> *Let $A \subseteq \mathbb{N}$. A is c.e. if and only if there exists a partial computable function f such that $A = \mathit{range}(f)$.*

Proof: ($\Rightarrow$) Suppose A is c.e.. Then either $A = \emptyset$ or $A = \text{range}(f)$ for some *computable* function f. If $A = \emptyset$, then A is the range of the empty partial computable function (always undefined). Otherwise $A = \text{range}(f)$ (and f is partial computable since it is computable).

($\Leftarrow$) Let $A = \text{range}(f)$ for some partial computable f. Then $f = \varphi_e$ for some $e \in \mathbb{N}$ (since f can be emulated by a Turing machine). Define the relation

$$C(s, x, y) : \varphi_{e,s}(x) \downarrow = y.$$

Then

$$A = \{y \in \mathbb{N} : \exists(s, x)\, C(s, x, y)\}.$$

(Formally I should have used $C(\langle s, x\rangle, y)$ for the relation.)

Thanks for watching my video. For more information, please visit `sjorv.github.io` for a giveaway of two $GME shares.

If you'd like to cheat on the homework, please stay for office hours! :D

If not, then bye. ;-;
To help you on your homework, please try the following proof methods.

Common proof techniques

Proof by intimidation Trivial!

Proof by cumbersome notation The theorem follows immediately from the fact that $\left|\bigoplus_{k\in S}\left(\mathfrak{K}^{\mathbb{F}^{\alpha}(i)}\right)_{i\in\mathcal{U}_k}\right| \preccurlyeq \aleph_1$ when $[\mathfrak{H}]_{\mathcal{W}} \cap \mathbb{F}^{\alpha}(\mathbb{N}) \neq \emptyset$.

Proof by inaccessible literature The theorem is an easy corollary of a result proven in a hand-written note handed out during a lecture by the Yugoslavian Mathematical Society in 1973.

Proof by ghost reference The proof my be found on page 478 in a textbook which turns out to have 396 pages.

Circular argument Proposition 5.18 in [BL] is an easy corollary of Theorem 7.18 in [C], which is again based on Corollary 2.14 in [K]. This, on the other hand, is derived with reference to Proposition 5.18 in [BL].

Proof by authority My good colleague Andrew said he thought he might have come up with a proof of this a few years ago...

Internet reference For those interested, the result is shown on the web page of this book. Which unfortunately doesn't exist any more.

Proof by avoidance *Chapter 3:* The proof of this is delayed until Chapter 7 when we have developed the theory even further. *Chapter 7:* To make things easy, we only prove it for the case $z = 0$, but the general case in handled in Appendix C. *Appendix C:* The formal proof is beyond the scope of this book, but of course, our intuition knows this to be true.

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